Mathematics and the planning of public works in China, Han to Yuan

Donald B. Wagner
Nordic Institute of Asian Studies
University of Copenhagen

www.donwagner.dk/saw.pdf

JIUZHANG SUANSHU 九章算術
ARITHMETIC IN NINE CHAPTERS
1ST CENTURY CE?
Jiuzhang suanshu, ch. 5, problem 4

A dyke has breadth 2 zhang, upper breadth 8 chi, height 4 chi, and length 12 zhang 7 chi.
What is the volume?
Answer: 7112 [cubic] chi. \(\frac{(20+8) \times 4 \times 127}{2} = 7112 \text{ chi}^3\)

JZSS, 5: 4, continued

By the winter norm, one person’s labour is 444 [cubic] chi.
How many labourers are used?
Answer: 16 \(\frac{2}{111}\)
Method: . . .

\[
\frac{7112 \text{ chi}^3}{444 \text{ chi}^3 / \text{labourer}} = 16 \frac{2}{111} \text{ labourers}
\]
JZSS, 5: 21

A pool has upper breadth 6 zhang, length 8 zhang, lower breadth 4 zhang, length 6 zhang, and depth 2 zhang. What is the volume?
Answer: 70,666 $^2/3$ [cubic] chi.

JZSS 5: 21, continued

The carrying of the earth is 70 paces, with 20 paces up and down wooden steps. Two [paces] on the steps correspond to five on a level path. For resting time, one is added for each ten. Time for loading and unloading is equivalent to 30 paces. One round trip is determined to be 140 paces. The capacity of a basket of earth is 1 chi 6 cun [i.e. 1.6 cubic chi]. The autumn norm for one person’s labour is equivalent to walking 59½ li [ca. 22 km].

What is the volume [of earth] carried by each person, and how many labourers are used?

Answer: One person carries 240 [cubic] chi, and the number of labourers is $346 ^{62}/_{153}$.

Method: • • •
Digging a canal, ca. 1840

Fig. 8–6. The Chinese genius for organizing very large numbers of workers in civil engineering operations illustrated by a unique drawing from the autobiography of the high official and hydraulic engineer Lin Ching (Hang Hauh Yin Yuane Thu Chi, ch. 3, pp. 81a, b, 82a).

Building the Red Flag Canal, 1958–65

http://www.youtube.com/watch?v=wtkijk9nGBM
XU SHANG 許商, 1ST CENTURY BCE

Xu Shang 許商, 1st century BCE

- Held numerous high positions in government (Han shu, 19b: 836, 841–2).

- Wrote a book in one chapter on natural phenomena, Wu xing zhuanji 五行傳記 (Han shu, 30: 1705). Possibly concerned with astronomy or the calendar (Han shu buzhu, 88: 13a).

- Wrote a book in 26 chapters on calendrical and astronomical calculations, Xu Shang suanshu 許商算術 (Han shu, 30: 1766).

(See also Loewe, Biographical dictionary, p. 622.)
Xu Shang’s story – background

• (About 100 BCE?) the Yellow River had bifurcated, forming a tributary which was called the Tunshi River.

• In 39 BCE the Tunshi River silted up.

• In 32 BCE Feng Qun 馮逡 reported that the Yellow River was in danger of overflowing its dykes, and that dredging the Tunshi River would ease the pressure and reduce the danger of flooding.

Han shu, 9: 1686–1699 (See also Needham, Science and civilisation in China, 4.3: 329–331).

Xu Shang gives advice –

“The Emperor passed the memorial to the Chancellor and the Imperial Counsellor. They responded that Xu Shang was an authority on the Book of Documents, that he was good at calculating, and that he could estimate the labour required. He was sent to inspect the situation. He reported that the Tunshi River was the cause of flooding, but that the local labour resources were not sufficient, and dredging could be postponed.”
– unfortunate advice

- Three years later the Yellow River overflowed its dykes and caused devastation in four commanderies and 32 prefectures.
- Wang Yanshi 王延世 was sent, and he repaired the dyke with stone-filled gabions.
The Yellow River Delta

Gabions

Fig. 913. A weir of gabions near Chhêngtu in Szechuan (orig. photo., 1943). The plaited bamboo cylinders filled with stones can be built into any desired formation; in structures such as this the successive layers are covered with bamboo matting. Cf. pp. 295, 321 ff., 339 ff. and Figs. 895, 914.
More gabions

Fig. 8—Closing the final gap in the 5000-foot-wide break at Huayuankou by use of long willow-stone “sausages.”

XU SHANG 許商, CONTINUED
Problems with the experts

• Two years later the Yellow River again overflowed its dykes. It was proposed that Wang Yanshi should again be sent, but Du Qin 杜欽 objected that another expert, Yang Yan 楊焉, should be sent, together with Xu Shang and Shengma Yannian 乘馬延年.

“Wang Yanshi and Yang Yan will certainly have violent disputes; there will be deep discussions and mutual criticism. Xu Shang and Shengma Yannian both understand calculation and can estimate labour requirements, so that they can distinguish truth from error. They can choose the correct plan and follow it, so that there surely will be success.”

• It was a success, and Wang Yannian was rewarded with 600 jin 斤 of gold (ca. 150 kg).

ZU GENG 祖暅, 5TH CENTURY CE
Zu Geng 祖暅, 5th century CE

- Son of the mathematician Zu Chongzhi 祖冲之.
- Used a version of Cavalieri’s Theorem.
- Derived the correct formula for the volume of a sphere.
- Wrote something about astronomy.

Zu Geng quoted in *Kaiyuan zhanjing* 開源占經

“This statement by Jiang Ji is wrong: ‘Stars are like the moon, receiving [light] from the sun and only then being visible.’ If stars were inside the sun [?], they would necessarily have phases; this is not the case. It is well known that stars • • • have a constant brightness. Thus the bodies of stars have their own light and do not receive [light] from the sun before they shine. • • •”
Zu Geng gives unwanted advice ...

*(Liang shu 梁書, 20:291–2; Nan shi 南史, 45: 1374–5)*

• About 500 CE, Wang Zu 王足 presented calculations and proposed that the Huai River 淮水 should be dammed to provide irrigation for Shouyang 壽陽.

“The Emperor Gaozu 高祖 was in favour of this, and sent the waterworks master Chen Chengbo 陳承伯 and the Construction Supervisor [*caiguan jiangjun 材官將軍*] Zu Geng to inspect the topography. They both reported that the sandy mud of the Huai River was too light and insubstantial, so that the project could not be completed.”

• The Emperor did not accept this, and set some 200,000 labourers to work building the dam.

... and ends up in prison.

• The dam was to be closed in 515, but various difficulties were encountered, the people were distressed, and many labourers died. It was finally closed in 516, and was a great success.

• However, later it was not properly maintained.

“In the autumn, in the 8th month, the Huai River rose disastrously, the dam broke in many places, and the river flowed violently to the sea. Zu Geng was charged and sent to prison.”
WANG XIAOTONG 王孝通 (6TH–7TH CENT. CE)

Wang Xiaotong 王孝通 (6th–7th cent. CE)

• Served the Sui and Tang dynasties in posts concerned with calendrical calculations.
• Wrote *Jigu suanjing* 緝古算經, “Continuation of ancient mathematics”.
  – One simple pursuit problem, stated as an astronomical problem.
  – 13 problems in solid geometry, stated as construction problems.
  – 6 problems in plane geometry, completely abstract.
• In 656 it was made one of the official “canons” for mathematical education.
**Jigu suanjing, problem 3**

Suppose a dyke is to be built. The difference between the upper and lower widths of the west end is 6 *zhang* 8 *chi* 2 *cun*, the difference between the upper and lower widths of the east end is 6 *chi* 2 *cun*, the height of the eastern end is 3 *zhang* 1 *chi* less than the height of the western end, the upper width is 4 *chi* 9 *cun* greater than the height of the eastern end, the straight length is 476 *chi* 9 *cun* greater than the height of the eastern end.

![Diagram of a dyke](image)

**Jigu suanjing, problem 3, continued**

- $b_2 - a_2 = 682$ *cun*
- $b_1 - a_1 = 62$ *cun*
- $h_2 - h_1 = 310$ *cun*
- $a - h_1 = 49$ *cun*
- $l - h_1 = 4769$ *cun*

$$h_1^3 + 5004 h_1^2 = 1,169,953\frac{3}{5} \text{ cun}^3$$

$$h_1 = 41,107,188\frac{3}{5} \text{ cun}$$
Jigu suanjing, problem 3, continued


People in former times, carrying 2 dou 4 sheng 8 ge [= 248 ge] of soil on their backs and travelling 192 bu on a level road, did 62 trips in one day. In the present situation there are hills to climb and rivers to cross to obtain the soil: there are only 11 bu of level road, the slanted height of the hill is 30 bu, and the width of the river is 12 bu. When climbing a hill 3 [bu] is equivalent to 4 [bu of even road], when descending a hill 6 [bu] is equivalent to 5 [bu], and when crossing water 1 [bu] is equivalent to 2 [bu]. • • •

b₂ – a₂ = 682 cun
b₁ – a₁ = 62 cun
h₂ – h₁ = 310 cun
a – h₁ = 49 cun
l – h₁ = 4769 cun

h₁³ + 5004 h₁² = 1,169,953½ cun³
h₁ = 41,107,188½ cun
Jigu suanjing, problem 3, continued

Jigu suanjing, problem 3, continued
Problem from a modern textbook for engineers

What relation between the height and the diameter of a tin can minimizes the amount of material used for a given volume?

Answer: \( \text{height} = \text{diameter} \).
Problem from a modern textbook for engineers:

Problem: A cylindrical soap solution tank is being filled at the rate of 10 cubic meters per minute. At what rate is the height of the solution increasing when the amount of solution is 500 cubic meters?

Answer: height increases at 0.375 meters per minute.
GUO SHOUJING 郭守敬, 1231–1316

Nathan Sivin

Granting the Seasons

The Chinese Astronomical Reform of 1276: With a Study of Its Many Dimensions and an Annotated Translation of Its Records

Springer
Guo Shoujing 郭守敬, 1231–1316
(From the biography by Ho Peng Yoke in de Rachewiltz et al. In the service of the Khan, 1993, pp. 282–299.)

- His grandfather, Guo Rong, was well-educated in the classics and mathematics, and had an interest in water-works.

- In Xingzhou in 1251, when he was 20, the Pacification Commissioners, informed of Guo Shoujing’s talents, had him produce plans and cost estimates for a bridge, and put him in charge • • •

- In 1262 Guo Shoujing presented six different schemes for improving irrigation and waterways in the region south of Yanjing and north of the Yellow River. • • • He was immediately appointed Superintendent of Waterways. The first of his suggestions was actually implemented.

Guo Shoujing, continued

- In 1271 Guo Shoujing was made Director of Waterways.

- In 1275 he was ordered to survey the plain at the confluence of the Yellow and Huai Rivers with a view to building ports and waterways • • •.

- In 1291 he presented a scheme for improving transport on the Grand Canal • • •

- In 1298 he was summoned to plan a new canal • • •. He advised that the canal should be at least 77–107 metres wide to carry the summer surge. Senior officials, wishing to save costs, reduced the width by 1/3. The canal broke its banks in 1299, causing great loss of life and property.
Hefang tongyi, mathematical chapter, Problem 22.

Suppose there is a pit to be levelled with fascines. The length of the pit is 180 bu. The breadth is 20 bu, and the depth is 1 zhang 5 chi. For every 3 chi 6 cun [i.e., 3.6 cubic chi], one bundle of twigs and straw is used, using half of each. How much of each is used?

Answer: 1875 bundles.

Method: • • •

\[
\frac{(180 \text{ bu} \times 5 \text{ chi/bu}) \times (20 \text{ bu} \times 5 \text{ chi/bu}) \times 15 \text{ chi}}{3.6 \text{ chi}^3 / \text{ bundle}} \times \frac{1}{2} = 1875 \text{ bundles}
\]
Fascines on a German canal

Into a Dike Breach Near Lianocheng Goes a Giant Bale of Kuoliang

When it is in place, a foot of earth will be placed on top to make it settle deeper into the mud at the bottom of the Hoang Ho. This breach occurred during the disastrous flood of 1935, when 6,000 square miles of western Shantung farmland were inundated.
**Hefang tongyi, Problem 19**

Suppose an ‘earthen ox’ [tu niu 土牛] is to be built. The upper length is 2 zhang 5 chi, the lower length is 3 zhang 5 chi, the upper breadth is 1 zhang, and the lower breadth is 2 zhang. The height is 1 zhang 8 chi. What is the volume?


Method: • • •

**Hefang tongyi, Problem 19, continued**

\[
\left(\frac{25 \text{ chi} + 35 \text{ chi}}{2}\right) \times \left(\frac{10 \text{ chi} + 20 \text{ chi}}{2}\right) \times 18 \text{ chi} = 8100 \text{ chi}^3
\]

Correct calculation:

\[
\left(\frac{2 \times 25 \text{ chi} + 35 \text{ chi}}{6}\right) \times 10 \text{ chi} + \left(\frac{2 \times 35 \text{ chi} + 25 \text{ chi}}{6}\right) \times 18 \text{ chi} = 8250 \text{ chi}^3
\]
Hefang tongyi, Problem 18

Suppose a square terrace is to be built. Above it is 3 zhang 2 chi square, below it is 5 zhang 6 chi square, and the height is 48 chi. What is the volume?


Method: • • •

\[
\frac{\left[(32 \text{ chi})^2 + (56 \text{ chi})^2 + (32 \text{ chi} \times 56 \text{ chi})\right] \times 48 \text{ chi}}{36} = 7936 \text{ chi}^3
\]

Correct calculation:

\[
\frac{\left[(32 \text{ chi})^2 + (56 \text{ chi})^2 + (32 \text{ chi} \times 56 \text{ chi})\right] \times 48 \text{ chi}}{3} = 95,232 \text{ chi}^3
\]