

Doubts concerning the attribution of Liu Hui's
commentary on the *Chiu-chang suan-shu*

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DOUBTS CONCERNING THE ATTRIBUTION
OF LUI HUI'S COMMENTARY ON THE
CHIU-CHANG SUAN-SHU

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1. Introduction

In studying the commentary on the *Chiu-chang suan-shu* 九章算術 ('Arithmetic in nine chapters') attributed to Liu Hui 劉徽 (third century AD), I have often been struck by certain curious breaks in the style of the exposition. It seemed to me possible that the commentary as we have it today is a conflation of two or more commentaries. Arguments based on such impressionistic criteria as "the style of the exposition" are difficult and dangerous, and I do not propose to say more about such matters here. However I have found a certain amount of more concrete evidence in favor of this hypothesis, and feel it may be useful to present the evidence here.

On Liu Hui himself and his book we have only the tiniest scraps of information. The *Ssu-k'u ch'üan-shu tsung-mu t'i-yao* 四庫全書總目提要 states the available information succinctly:¹

The *Chin shu* 晉書 refers to "Liu Hui's commentary on the *Chiu-chang*, of the 4th year of the *Ching-yüan* 景元 period of the Wei 魏." [= AD 263]. But the commentary

I am grateful to Albert Dien, for encouraging me to go deeper into this problem, and to Huy Dong Shin 申暉東, for writing the Chinese characters.

¹ Punct. ed., Shanghai 1933, vol. 2, p. 814.

refers to "the bronze *hu* 斛 in the armory of Chin 晉".

Thus Liu Hui made further revisions after the beginning of the Chin (AD 265).

Generally modern scholars, in discussing Liu Hui, either repeat this judgement or ignore the two mentions of the Chin dynasty in the commentary. As will be seen below, the situation is a good deal more complex.

All extant versions of the *Chiu-chang suan-shu* derive ultimately from an edition by Li Ch'un-feng 李淳風 (602–670) in his *Suan-ching shih-shu* 算經十書 ('Ten mathematical classics'). Here the text is given in large characters, the commentary attributed to Liu Hui in small characters, and Li Ch'un-feng's commentary in small characters preceded by the words *Ch'en Ch'un-feng teng chin an* 臣淳風等謹按, 'The ministers [Li] Ch'un-feng and others respectfully comment'.

2. Quotations in the *Sui shu* and *Chin shu*

The *Lü-li chih* 律曆志 ('Treatise on the pitch-pipes and the calendar') chapters of both the *Sui shu* 隋書 and the *Chin shu* 晉書 were written by Li Ch'un-feng and revised by others after him.² Since the material discussed below occurs in both, it is probably a safe assumption that it comes from his original versions.

References to Liu Hui's commentary occur in sections of the *Chin shu* and *Sui shu Lü-li chih* concerning the history of linear measures³ and of grain measures.⁴ In each case the section in the

² *Chiu T'ung shu* 舊唐書 (punct. ed., Shanghai 1975), p. 2718; *Sui shu* 隋書 (punct. ed., Peking 1973), p. 1903.

³ *Shen tu* 審度, *Chin shu* 晉書 (punct. ed., Peking 1974), pp. 490–491; *Sui shu*, pp. 401–408.

⁴ *Chia liang* 嘉量, *Chin shu*, p. 492; *Sui shu*, pp. 408–411.

Chin shu is a severely abridged version of the corresponding section in the *Sui shu*, but the parts in which Liu Hui's commentary is cited are virtually identical.

In the sections on grain measures a passage from Liu Hui's commentary is quoted in which he describes the standard *hu* 斛 of the "present" Grand Superintendent of Agriculture and the *hu* of Wang Mang 王莽.⁵ The quotation omits a few superfluous words but otherwise corresponds exactly to the present text of Liu Hui's commentary.

In the sections on linear measures the same passage of Liu Hui is used to show that the *ch'ih* 尺 of the time of Wang Mang was equal to 0.955 *ch'ih* of the Wei.⁶ Here the passage is not quoted but briefly summarized.

There appears to be no way of determining how Li Ch'un-feng arrived at the date 263 for Liu Hui's commentary. The present edition contains no date. However we may note that Li Ch'un-feng was in a position to know a great deal about the history of the book. It is therefore fairly safe to conclude that the date 263 applies to some part of the text of Liu Hui's commentary as we have it today, and that Liu Hui was indeed the author of some part of the commentary.

3. Mentions of the Chin dynasty in Liu Hui's commentary

There are two passages in Liu Hui's commentary which mention the Chin dynasty. Both are discussions of Wang Mang's standard grain measures. There is also a third passage on this subject, and this is the passage cited in the *Chin shu* and *Sui shu*. Before considering these passages it is necessary to discuss in general what we know of these grain measures.

⁵ *Chiu-chang suan-shu* 九章算術 (punct. ed. in Ch'ien Pao-tsung 錢寶琮, ed., *Suan-ching shih-shu* 算經十書 Peking 1963, pp. 93–258), p. 176, lines 1–3.

⁶ *Chin shu*, p. 491; *Sui shu*, p. 404.

3.1. *The standard grain measures of Wang Mang*

The *Chia-liang hu* 嘉量斛 is a bronze vessel which includes five compartments intended to serve as standards for five units of grain measurement:

$$1 \text{ hu } 斛 = 10 \text{ tou } 斗 = 100 \text{ sheng } 升 = 1000 \text{ ko } 合 = 2000 \text{ yüeh } 龠$$

One example of this vessel is still in existence today, and has been studied by a number of modern scholars. The most convenient reference is an article by Erwin Reifler.⁷ This article gives a good description of the vessel, gives the text of five of its six inscriptions, with translations, and refers to a large number of primary and secondary sources. (The article's conclusions are outrageous, and should be ignored.)

As will be seen below (section 3.3), the inscriptions give the dimensions of the various compartments in a peculiar form. It seems to be an imitation of the description of a standard grain measure given in the *K'ao-kung-chi* 考工記 section of the *Chou li* 周禮.⁸

3.2. *Liu Hui's discussion of ancient grain measures*

In problem 25 of chapter 5 of the *Chiu-chang suan-shu* different volumes are given for a *hu* of three different commodities respectively:

- 1 *hu* of millet = 2.7 cubic *ch'ih*
- 1 *hu* of rice = 1.62 cubic *ch'ih*
- 1 *hu* of beans, peas, hemp, or wheat = 2.43 cubic *ch'ih*

⁷ Erwin Reifler, "The Philological and Mathematical Problems of Wang Mang's Standard Grain Measures. The Earliest Chinese Approximation to π ", *Symposium in Honor of Dr. Li Chi on his Seventieth Birthday*, Part 1 慶祝李濟先生七十歲論文集上冊 (Taipei 1965), pp. 387-402.

⁸ *Shih-san ching chu-shu* 十三經注疏 (punct. ed., Shanghai 1935), vol. 1, pp. 916-917.

These ratios stand in the ratio 50:30:45; these are the ratios given for the exchange of these commodities in chapter 2, which deals with ratios.

Liu Hui's comment says:⁹

These are in the ratio of the degree of refinement, and do not correspond to the measuring vessel. Five parts of millet correspond to three of rice; therefore a *hu* of rice is $\frac{3}{5}$ of a *hu* of millet, and beans, peas, hemp, and wheat are treated according to their basic ratio. Therefore if these are treated as measuring vessels, none corresponds to the modern *hu*.

[The following is the passage quoted in the *Chin shu* and *Sui shu*.]

The *hu* of the present Grand Superintendent of Agriculture is round, with diameter 1.355 *ch'ih* and depth 1 *ch'ih*. Using [Liu] Hui's [i.e. 'my'] method [i.e. the calculation of the volume of a cylinder using $\pi = 3.14$], the volume is 1441 [cubic] *ts'un* 寸 [10 *ts'un* - 1 *ch'ih*], and if the fraction is included there is an additional $\frac{3}{10}$ [cubic] *ts'un*.

The bronze *hu* of Wang Mang, with the present value of the *ch'ih*, has depth 0.955 *ch'ih* and diameter 1.3687 *ch'ih*. Calculated by [Liu] Hui's method, and using the present value of the *hu*, it holds 0.974 + *hu*. . . .

[Here follows a discussion of some units of measurement described in the *Chou li* 周禮 and the *Tso chuan* 左傳, not translated here.]

The bronze *hu* of Wang Mang is the same as the *hu* described in the *Lü li chih* of the *Han shu* 漢書.¹⁰

⁹ *Chiu-chang suan-shu*, p. 175, line 13-176, line 7.

¹⁰ The reference is to *Han shu* 漢書 (punct. ed., Peking 1975), p. 967. Here a brief description is given, with no mention of the inscription and with insufficient technical detail.

This comment contains the passage quoted in the *Chin shu* and *Sui shu*, and there is nothing in it which could not have been written in AD 263, so we can conclude that it was written by Liu Hui. (The last sentence could be a sub-comment by someone else on Liu Hui's comment.)

Note here that Liu Hui does not mention the inscription on the *hu*. This inscription gives the volume as 1620 cubic *ts'm* (= 1.62 cubic *ch'ih*); since this is the same as that given in the *Chiu-chang suan-shu* for a *hu* of rice, one might suppose that he would have been interested. There is at any rate no evidence here that Liu Hui knew of the inscriptions; it is more likely that he had some written description of the *hu* which gave its dimensions.

3.3 A more detailed discussion of the *hu* of Wang Mang, usually attributed to Liu Hui

Problem 28 of chapter 5 of the *Chiu-chang suan-shu* concerns the calculation of the circumference of a cylindrical granary given its height and its capacity in *hu* of rice. The text does not specifically mention the conversion from *hu* to cubic *ch'ih*, presumably because this is covered in problem 25 (see section 3.2 above), but it is clear from the answer given that the value used is 1.62 cubic *ch'ih* per *hu*.

The first part of the comment concerns the calculation of the circumference from the volume in cubic *ch'ih*, and there is no obvious reason to suppose that this was not written by Liu Hui. Then comes the following discussion of the Wang Mang grain measures:¹¹

In the armory of Chin there (is/was) a bronze *hu* made by Wang Mang in the Han period. The inscription in seal script on the side of the *hu* says:

“Standard *Chia-liang hu*. If a square of 1 *ch'ih* is inscribed inside a circle, the excess¹² is 9 *li* 釐 5 *hao* 毫 [− 0.0095

¹¹ *Chiu-chang suan-shu*, p. 178, lines 2–6.

¹² *T'iao-p'ang* 庹旁.

ch'ih], and the area is 162 [square] *ts'un*. The depth is 1 *ch'ih*, and the volume is 1620 [cubic] *ts'un*. It holds 10 *tou*.”

[Next the commentary quotes the inscription on the *tou*. It has the same form as the inscription above, and is not translated here.]

The *sheng*, *ko*, and *yüeh* all have inscriptions. The *sheng* is placed on the side of the *hu*, and the *ko* and *yüeh* are on the ear of the *hu*. On the back there is an appreciation.¹³ It is the same as that in the present *Lü li chih*, and is also that commonly used in the Wei 魏 and Chin 晉 periods.

Here I have roughly explained the inscriptions and measurements of Wang Mang's bronze *hu*, but the inscriptions on the *sheng*, *ko*, and *yüeh*¹⁴ were not completely obtained.

Here follows an explanation of the method of the *Chiu-chang suan-shu*, preceded by the words *an tz'u shu* 按此術, 'comment on this method'. There is no obvious relationship with the passage translated above.

If the author of this passage lived in the Chin period, it would be very odd for him to refer to the dynasty as he does. He would be more likely to use some such expression as *Ta Chin* 大晉 or *Huang ch'ao* 皇朝. He would be even more likely simply to write *chin* 今, or not to give any indication of time at all. The opening sentence looks much more like a reference by someone after the Chin period to a written source.

There is however a more fundamental reason for concluding that Liu Hui did not write this passage. Ch'ien Pao-tung punctuates in such a way that the sentence, “It is the same as that in the present *Lü li chih*, and is also that commonly used in the Wei and Chin periods”, refers to the appreciation inscribed on the

¹³ *Tsan-men* 讚文.

¹⁴ Reading *yüeh* 餗 for *cho* 勺.

back of the *hu*. But this is a highly unlikely interpretation, for the appreciation is not something which could be “commonly used”. The sentence must refer to the standard grain measure itself. But Liu Hui states in the passage translated in section 3.2 above that the Wang Mang *hu* was *different* from the *hu* of his own time.

3.4. A discussion of the value of π implied by the inscription on the Wang Mang *hu*, sometimes attributed to Liu Hui, but probably by Tsu Ch’ung-chih

Problem 31 of chapter 1 of the *Chiu-chang suan-shu* gives the area of a circle as



$$A = \frac{c}{2} \frac{d}{2}$$

where c is the circumference and d the diameter. A long comment follows, which may be divided up as follows:¹⁵

1. Proof that the formula is correct.
2. Calculation of $\pi \approx 3.14$.
3. Discussion of the inscription on the Wang Mang *hu*, mentioning the Chin dynasty.
4. Continuation of part 2, giving the result $\pi \approx 3.1416$.
5. Discussion of how to improve the calculation even further.
6. Further comment by Li Ch’un-feng.

In part 6 Li Ch’un-feng states:

Although [Liu] Hui gives (this one/these two) method(s),

[Some extant versions have *i* , ‘one’, for *erh* , ‘two’.]

he was still not able to achieve great precision. Tsu Ch’ung-chih 祖冲之 [429–500], because of this lack of precision, especially recalculated the numbers [i.e. the value of π].

¹⁵ *Chiu-chang suan-shu*, (1) p. 103, line 10–104, line 5; (2) p. 104, line 5–106, line 2; (3) p. 106, lines 2–3; (4) p. 106, lines 3–8; (5) p. 106, lines 8–9; (6) p. 106, lines 9–17.

The present editors have collected [the calculations of] various scholars and researched their correctness; [the results of] [Tsu] Ch’ung-chih are the most precise. Therefore it [i.e. Tsu Ch’ung-chih’s calculation] is appended to [Liu] Hui’s method(s).

Thus some part of the comment is by Tsu Ch’ung-chih, but there is no obvious indication of where it begins. Virtually none of Tsu Ch’ung-chih’s mathematical writing is now extant, so there is no straightforward way of determining where the break was.

Part 2 begins with the words *chin an* 謹按. ‘I respectfully comment’. This phrase is normally used only in comments which in a formal sense are addressed to the Emperor. The phrase does not occur elsewhere in Liu Hui’s commentary, but it is part of the normal prefix to Li Ch’un-feng’s comments, as noted in section 1 above. Thus this might be the mark indicating the beginning of the quotation from Tsu Ch’ung-chih. However under this assumption we have no calculation of π by Liu Hui, and this seems to conflict with the remarks of Li Ch’un-feng translated above.

Throughout his commentary Liu Hui uses the value $\pi \approx 3.14$, so it makes sense to suppose that part 2 was written by him. The Ch’ing commentator Li Huang 李潢 said he “suspected” that the quotation from Tsu Ch’ung-chih began with part 3.¹⁶

Li Yen 李儼 assumes that part 4 is by Tsu Ch’ung-chih,¹⁷ but appears not to take any stand on part 3.

Ch’ien Pao-tsung takes parts 1–4 to be by Liu Hui, and only part 5 to be by Tsu Ch’ung-chih.¹⁸ This assumption appears to conflict with the remarks of Li Ch’un-feng.

¹⁶ *Chiu-chang suan-shu hsi-ts’ao t’u-shuo* 九章算術細草圖說, ed. by Li Huang 李潢 (Yü-hung T’ang ed., 1820), *chüan* 1, p. 35a.

¹⁷ Li Yen 李儼, *Chung-kuo ku-tai shu-hsüeh shih-liao* 中國古代數學史料 (2d ed., Shanghai 1963), pp. 54–55.

¹⁸ *Suan-ching shih-shu*. p. 107. n. 7.

I personally place most emphasis on the prefix *chin an* in part 2; thus the original edition may have contained a calculation of π by Liu Hui which has since been lost, and the quotation from Tsu Ch'ung-chih probably begins with part 2. For our present purposes it is sufficient to note that part 3, which contains a reference to the Chin dynasty, is not necessarily by Liu Hui.

Part 3 quotes the inscription on the *hu* and notes that it is fairly consistent with the value $\pi \approx 3.14$. We know that Tsu Ch'ung-chih wrote something on the value of π implied by the Wang Mang *hu*: it is mentioned in the *Lü li chih* chapter of the *Sui shu*.¹⁹ See also the remark by Tsu Ch'ung-chih translated in section 5 below.

4. A comment on the volume of a sphere, usually attributed to Liu Hui

Problems 23 and 24 of chapter 4 of the *Chiu-chang suan-shu* concern the calculation of the diameter of a sphere when the volume is given. The calculation is given as

$$d = \sqrt[3]{\frac{16}{9}V}.$$

Since the true formula is

$$d = \sqrt[3]{\frac{6}{\pi}V},$$

it is sometimes suggested that the *Chiu-chang suan-shu* here used the value $\pi \approx 3\frac{3}{8}$. This is unlikely, since everywhere else in the *Chiu-chang suan-shu* the value $\pi \approx 3$ is used.

There is a long comment on this calculation which may be divided up as follows:²⁰

1. It is known that the volume of a cylinder inscribed in a cube is $\frac{\pi}{4}$ times the volume of the cube. The *Chiu-chang*

¹⁹ *Sui shu*, p. 388.

²⁰ *Chiu-chang suan-shu*, (1) p. 155, line 5–156, line 2; (2) p. 156, lines 2–5; (3) p. 156, lines 5–7; (4) p. 156, line 7–157, line 4; (5) p. 157, lines 4–5; (5.1) p. 157, line 5–158, line 8; (5.2) p. 158 line 8; (5.3) p. 158 lines 8–10.

suan-shu uses the value $\pi \approx 3$: therefore the above formula implies that the volume of a sphere inscribed in the same cube is $\frac{\pi}{4}$ times the volume of the cylinder.

2. "But this way of thinking is incorrect." Proof that the volume of the sphere is less than $\frac{\pi}{4}$ times the volume of the cylinder.
3. The commentator, having proved that the given formula is incorrect, is unable to find the correct formula. A poem on the frustrations of the mathematician, ending, "I dare to let the doubtful points stand, / Waiting for one who can expound them."
4. A long discussion of Chang Heng's 張衡 (78–135) treatment of the volume of a sphere.
5. Comment by Li Ch'un-feng. "Tsu Keng-chih 祖暅之 [Tsu Ch'ung-chih's son] stated that both Liu Hui and Chang Heng took the cylinder to have the proportion of a square and the sphere to have the proportion of a circle, and that he therefore established a new method. Tsu Keng-chih's 'Extraction of the spherical root'²¹ says:"

- 5.1. Proof by Tsu Keng-chih that $d = \sqrt[3]{\frac{6}{\pi}V}$.

- 5.2. A poem on the triumph of the mathematician: "The proportions are extremely precise, / And my heart shines. / Chang Heng copied the ancient, / Smiling on posterity; / Liu Hui followed the ancient, / having no time to revise it. / Now what is so difficult about it? / One need only think."

- 5.3. Tsu Keng-chih's formula again, using the value $\pi \approx 3\frac{1}{7}$.

Parts 1–4 have always been attributed to Liu Hui. Part 5.2 is probably by Tsu Keng-chih, and part 5.3 is probably by Li Ch'un-feng; there is no indication of where the quotation from Tsu Keng-chih ends.

²¹ *K'ai li-yüan shu* 開立圓術.

Did Liu Hui really write part 2? This seems very unlikely. It is specifically stated in parts 5 and 5.2 that Liu Hui accepted the assumption that the volume of the sphere is π times the volume of the cylinder; part 2 is a proof that this assumption is false. It is unlikely that Tsu Keng-chih had not seen the text of part 2: part 5.1 uses the same general method as part 2, and the poem of part 5.2 appears to be an answer to part 3.

5. Other commentators on the *Chiu-chang suan-shu*

Given the suspicion that the commentary attributed to Liu Hui may actually be a conflation of two or more commentaries, we must now consider who might have written these other commentaries.

The obvious place to look first is in the bibliographical chapter of the *Sui shu*. This bibliography was compiled in Li Ch'un-feng's time, by a group with which he must have had some contact. Furthermore this bibliography, unlike many others, explicitly includes only books which were extant at the time it was compiled.²² It lists a number of editions of the *Chiu-chang suan-shu*; among these are the following.²³

Chiu-chang-suan-shu, 10 *chüan* 卷, by Liu Hui.

Chiu-chang suan-shu, 2 *chüan*, further explicated by Hsü Yüeh 徐岳 and Chen Luan 甄鸞.

Chiu-chang suan-shu, 1 *chüan*, additional commentary (*shu* 疏) by Li Tsun-i 李遵義.

Chiu-chang suan-ching 經, 25 *chüan*, by Hsü Yüeh, Chen Luan, and others.

Chiu-chang suan-ching, 2 *chüan*, commentary by Hsü Yüeh.

Hsü Yüeh was active in the Han and Wei periods, and was famous as a mathematician and astronomer.²⁴

²² *Sui shu*, p. 908.

²³ *Sui shu*, p. 1025.

²⁴ Li Yen, pp. 51-52.

Chen Luan wrote a number of books on mathematics and astronomy, but he is better known as a Taoist apologist. His book *Hsiao-tao lun* 笑道論 is dated AD 570.²⁵ Li Ch'un-feng's edition of the *Chou-pei suan-ching* 周髀算經 includes a commentary by Chen Luan, and it is therefore quite plausible that Li Ch'un-feng would have included Chen Luan's commentary in his edition of the *Chiu-chang suan-shu*. On the other hand Chen Luan's commentary on the *Chou-pei suan-ching* is very uninteresting: it simply gives the numerical working for each calculation. There is almost nothing of this sort in Liu Hui's commentary on the *Chiu-chang suan-shu*.

I have not found any further information on Li Tsun-i. There were two persons named Li Tsun 李遵 in the fourth and fifth centuries; the available information on them gives no indication that they were interested in mathematics.²⁶

Tsu Ch'ung-chih's biography states that he wrote a commentary on the *Chiu-chang suan-shu*.²⁷ No edition by him is mentioned in the *Sui shu* bibliography, but several editions are listed without the editor's name. (Tsu Ch'ung-chih's *Chui shu* is listed with no mention of his name.²⁸)

There is solid evidence that Tsu Ch'ung-chih wrote something on precisely the same subjects as the passages I have cast doubt on above. In a memorial submitted in or shortly after 462, answering a criticism that his *Ta-ming Calendar* 大明曆 did not accord with ancient practice, he wrote:²⁹

²⁵ Li Yen, pp. 74-75; E. Zürcher, *The Buddhist Conquest of China* (Leiden 1959), pp. 296, 302, 305.

²⁶ *Wei shu* 魏書 (punct. ed., Peking 1974), p. 895; *Pei Ch'i shu* 北齊書 (punct. ed., Peking 1972), pp. 392-394.

²⁷ *Nan Ch'i shu* 南齊書 (punct. ed., Peking 1972), p. 906; *Nan shih* 南史 (punct. ed., Peking 1975), p. 1774.

²⁸ *Sui shu*, p. 1025.

²⁹ *Sung shu* 宋書 (punct. ed., Peking 1974), p. 300.

In the case of the old error of the sphere, Chang Heng repeated it and did not correct it. In the case of the inscription on the Han period *hu*, Liu Hsin calculated the dimensions incorrectly. These are shortcomings of mathematicians. . . .

Chang Heng, K'an Tse 關澤, Wang Fan 王蕃, and Liu Hui all inquired into the arts of numbers, and each made mistakes. Long ago, in my spare time, I wrote a correction of these errors; the arguments are very clear, and easy to investigate in detail. . . .

6. Conclusions

From the material presented above it can be seen that there are two passages in the commentary attributed to Liu Hui that he probably did not write.

The passage on the Wang Mang *hu* (section 3.3 above) mentions the Chin dynasty, and this conflicts with the date AD 263 given by Li Ch'un-feng in the *Chin shu* and *Sui shu*. It mentions the Chin dynasty in a way which would be strange for a person living under the Chin. And it states that this *hu* was the same as that used in the writer's time, contrary to a statement by Liu Hui.

The passage on the sphere (section 4, part 2, above) was considered by Tsu Keng-chih not to be written by Liu Hui.

It seems then that the commentary is a conflation of Liu Hui's with one or more others. There were numerous other commentaries available to Li Ch'un-feng when he compiled his edition, and it would be logical for him to include all of these which were of high quality. In particular there is good reason to believe that Tsu Ch'ung-chih wrote a commentary, and that this was included in Li Ch'un-feng's edition.

I hope that further research will shed more light on this problem.